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Context-Free Grammars for Several Triangular Arrays

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Abstract: In this paper, we present a unified grammatical interpretation of the numbers that satisfy a kind of four-term recurrence relation, including the Bell triangle, the coefficients of modified Hermite polynomials, and the Bessel polynomials. Additionally, as an application, a criterion for real zeros of row-generating polynomials is also presented.

Keywords: recurrence relations; grammars; real zeros; Bell triangular array

MSC: 05A05; 05A15

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Citation: Zhou, R.R.; Yeh, J.; Ren, F. Context-Free Grammars for Several Triangular Arrays . *Axioms* **2022**, *11*, 297. https://doi.org/10.3390/axioms11060297

Academic Editor: Wei-Shih Du

Received: 26 April 2022
Accepted: 10 June 2022
Published: 20 June 2022

1. Introduction

Let A denote an alphabet, the letters of which are considered as independent commutative indeterminates. Then, the context-free grammar G over A is defined as a set of replacement rules that substitute the letters in A with formal functions on A. The formal derivative D is a linear operator, which is defined relative to a context-free grammar G (see [1]). For example, for $A = \{u, v\}$ and $G = \{u \to uv, v \to v\}$, then $D(u) = uv, D^2(u) = u(v + v^2), D^n(u) = u\sum_{k=1}^{n} S(n,k)v^k$, where $S(n,k)$ is the Stirling number of the second kind, i.e., the number of ways to partition $[n]$ into k blocks.

In [2], Hao, Wang, and Yang presented a grammatical interpretation of the numbers $T(n,k)$ that satisfy the following three-term recurrence relation:

$$T(n,k) = (a_1n + a_2k + a_3)T(n-1,k) + (b_1n + b_2k + b_3)T(n-1,k-1).$$

Very recently, there is a large literature devoted to the numbers $t(n,k)$ that satisfy the following four-term recurrence relation (see [3–7]):

$$t_{n,k} = (a_1n + a_2k + a_3)t_{n-1,k} + (b_1n + b_2k + b_3)t_{n-1,k-1} + (c_1n + c_2k + c_3)t_{n-1,k-2}, \quad (1)$$

with $t_{0,0} = 1$ and $t_{n,k} = 0$, unless $0 \le k \le n$. For example, Ma [8] showed that if $G = \{x \to xy, y \to yz, z \to y^2\}$, then $D^n(x^2) = x^2\sum_{k=0}^{n} R(n+1,k)y^kz^{n-k}$, where $R(n,k)$ is the number of permutations in S_n with k alternating runs, and it satisfies the recurrence relation

$$R(n,k) = kR(n-1,k) + 2R(n-1,k-1) + (n-k)R(n-1,k-2)$$

with the initial conditions $R(1,0) = 1$ and $R(1,k) = 0$ for $k \ge 1$.

Let

$$a(n,k) = \sum_{i=0}^{n} S(n,i)\binom{i}{k}$$

for $0 \le k \le n$. Clearly, $a(n,k)$ is the number of set partitions of $\{1,2,\ldots,n\}$ in which exactly k of the blocks have been distinguished. The numbers $a(n,k)$ satisfy the recurrence relation

$$a(n+1,k) = a(n,k-1) + (k+1)a(n,k) + (k+1)a(n,k+1), \quad (2)$$

with $a(0,0) = 1$, $a(0,k) = 0$ for $k \neq 0$ (see [9,10]). The triangular array $\{a(n,k)\}_{n,k}$ is known as the classical Bell triangle and is given as follows:

$$\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 3 & 1 & & & \\ 5 & 10 & 6 & 1 & & \\ 15 & 37 & 31 & 10 & 1 & \\ \vdots & & & & & \ddots \end{pmatrix}.$$

It appears that $a(n,0) = \sum_{i=0}^{n} S(n,i) = B_n$, which implies that the first column of the triangle array is made up of the Bell numbers B_n. A natural question is whether there exists a grammatical interpretation of the numbers $a(n,k)$.

This paper is motivated by exploring the grammatical interpretation of the triangular array $\{B(n,k)\}_{0\leq k\leq n}$ that satisfies the following four-term recurrence relation

$$\begin{aligned} B(n+1,k) = (a_1 n + a_2 k + a_3)B(n,k-1) + (b_1 n + b_2 k + b_3)B(n,k) \\ +(k+1)cB(n,k+1), \end{aligned} \tag{3}$$

where a_i, b_i, and c are integers for $1 \leq i \leq 3$ with $B(0,0) = 1$ and $B(0,k) = 0$ if $k \neq 0$. In Section 2, we present grammatical interpretations of the triangular array $\{B(n,k)\}$. In Section 3, we present grammatical interpretations of several combinatorial sequences, including the Bell triangle, the modified Hermite polynomials, the Bessel polynomials, and so on. In Section 4, we show the result of the real-rootedness of row-generating functions for $\{B(n,k)\}$, and apply the proposed criteria to the Bell triangular array as an example.

2. Grammatical Interpretations of the Triangular Array $B(n,k)$

We now present the first main result of this paper.

Theorem 1. *Suppose that a_i, b_i, and c are integers for $1 \leq i \leq 3$. Let*

$$G = \{I \to (a_2+a_3)IX + b_3 IY; X \to (a_1+a_2)X^2 + (b_1+b_2)XY + cY^2; Y \to a_1 XY + b_1 Y^2\}.$$

Then, we have

$$D^n(I) = I \sum_{k\geq 0} B(n,k) X^k Y^{n-k}, \tag{4}$$

where the coefficients $B(n,k)$ satisfy the recurrence relation (3).

Proof. Note that $D(I) = (a_2 + a_3)IX + b_3 IY$. Suppose that (4) holds for n. Then, by induction, we obtain

$$\begin{aligned} D^{n+1}(I) = D\{D^n(I)\} = \sum_{k\geq 0} B(n,k) D(I) X^k Y^{n-k} \\ + \sum_{k\geq 0} B(n,k) I D(X^k) Y^{n-k} + \sum_{k\geq 0} B(n,k) I X^k D(Y^{n-k}). \end{aligned}$$

Applying the rules of G, we can derive

$$\begin{aligned} \sum_{k\geq 0} B(n,k) I (a_2 + a_3) X^{k+1} Y^{n-k} + \sum_{k\geq 0} B(n,k) I b_3 X^k Y^{n+1-k} \\ + \sum_{k\geq 0} B(n,k) k I X^{k-1} Y^{n-k} \{(a_1 + a_2)X^2 + (b_1 + b_2)XY + cY^2\} \\ + \sum_{k\geq 0} B(n,k)(n-k) I X^k Y^{n-k} \{a_1 X + b_1 Y\}. \end{aligned}$$

Collate and merge similar items

$$\sum_{k\geq 0} B(n,k)(a_2+a_3+k(a_1+a_2)+(n-k)a_1)IX^{k+1}Y^{n-k}$$
$$+\sum_{k\geq 0} B(n,k)((n-k)b_1+k(b_1+b_2)+b_3)IX^kY^{n+1-k}$$
$$+\sum_{k\geq 0} B(n,k)kcIX^{k-1}Y^{n-k+2}.$$

Extracting the coefficient of IX^kY^{n+1-k}, we obtain (3). This completes the proof. □

Along the same lines of the proof of Theorem 1, one can easily derive the following result.

Proposition 1. *Let*

$$G=\{I\to(a_2+a_3)IX+b_3IY;\ X\to(a_1+a_2)X^2+(b_1+b_2)XY+cY^2;\ Y\to dX^2+a_1XY+b_1Y^2\}.$$

Then, we have

$$D^n(I)=I\sum_{k\geq 0} M(n,k)X^kY^{n-k},$$

where $M(n,k)$ satisfy the following five-term recursive relation:

$$\begin{aligned} M(n+1,k) = {} & (n-k+2)dM(n,k-2)+(a_1n+a_2k+a_3)M(n,k-1) \\ & +(b_1n+b_2k+b_3)M(n,k)+(k+1)cM(n,k+1). \end{aligned} \tag{5}$$

where a_i, b_i, c, and d are integers for $1\leq i\leq 3$.

When $d=0$, the recurrence relation (5) is degenerated into (3).

3. Applications

3.1. The Bell Triangle

The Bell triangle was proposed by Aigner [9] to provide a characterization of the sequence of Bell numbers by means of the determinants of Hankel matrices. As a special case of Theorem 1, we now present a grammatical interpretations of the Bell triangle.

Proposition 2. *Let $G=\{I\to IX+IY; X\to XY+Y^2; Y\to 0\}$. Then, we have*

$$D^n(I)=I\sum_{k\geq 0} a(n,k)X^kY^{n-k}=IY^na_n(\frac{X}{Y}).$$

Note $D^n(X)=XY^n+Y^{n+1}$, $D^n(Y)=0$. From Leibniz's formula, we obtain the following corollary:

Corollary 1. *For $n\geq 0$, we have*

$$a_{n+1}(x)=\sum_{i=0}^{n}\binom{n}{k}a_k(x)(x+1)$$

Let $D^n(IX)=I\sum_{k=0}^{n+1} b(n+1,k)X^kY^{n+1-k}$. It is routine to verify that

$$b(n+2,k)=b(n+1,k-1)+(k+1)b(n+1,k)+(k+1)b(n+1,k+1),$$

with $b(1,1)=1$ and $b(1,k)=0$ when $k\neq 1$. Since $D^{n+1}(I)=D^n(IX)+D^n(IY)$, it follows that $a(n+1,k)=b(n+1,k)+a(n,k)$.

Note that $D^n(X) = Y^n(X+Y)$. Then,

$$b(n+1,k) = \sum_{i=0}^{n+1-k} \binom{n}{i+k-1} a(i+k-1,k-1) + \sum_{i=0}^{n-k} \binom{n}{i+k} a(i+k,k).$$

3.2. On the Coefficients of Modified Hermite Polynomials

The modified Hermite polynomials have the following form:

$$\begin{aligned}
h(0,x) &= 1 \\
h(1,x) &= x \\
h(2,x) &= x^2+1 \\
h(3,x) &= x^3+3x \\
h(4,x) &= x^4+6x^2+3 \\
h(5,x) &= x^5+10x^3+15x \\
h(6,x) &= x^6+15x^4+45x^2+15
\end{aligned}$$

If $n-k \geq 0$ is even, let

$$T(n,k) = \frac{n!}{2^{\frac{n-k}{2}}(\frac{n-k}{2})!k!}.$$

Otherwise, set $T(n,k) = 0$. It should be noted that the numbers $T(n,k)$ are the coefficients of the modified Hermite polynomials (see A099174 [11]) and

$$T(n+1,k) = T(n,k-1) + (k+1)T(n,k+1).$$

Using Theorem 1, we obtain the following proposition.

Proposition 3. *Let $G = \{I \to IX; X \to Y^2; Y \to 0\}$. Then, we have*

$$D^n(I) = I \sum_{k \geq 0} T(n,k) X^k Y^{n-k} = IY^n h(n, \frac{X}{Y}).$$

Note that $D^n(X) = 0$ $(n \geq 2)$. From Leibniz's formula, we obtain the following corollaries:

Corollary 2. *For $n \geq 0$, we have*

$$h(n+1,x) = xh(n,x) + nh(n-1,x)$$

Corollary 3. *For $n \geq k \geq 1$, we have*

$$T(n+1,k) = T(n,k-1) + nT(n-1,k)$$

3.3. The Bessel Polynomials

As a well-known orthogonal sequence of polynomials, the Bessel polynomials $y_n(x)$ were introduced by Krall and Frink in [12], which can be defined as the polynomial solutions of the second-order differential equation

$$x^2 \frac{d^2 y_n(x)}{dx^2} + 2(x+1)\frac{dy_n(x)}{dx} = n(n+1)y_n(x)$$

After that, the Bessel polynomials have been extensively studied and applied (see [13–15]). Moreover, the polynomials $y_n(x)$ can be generated by using the Rodrigues formula (see [11] [A001498]):

$$y_n(x) = \frac{1}{2^n} e^{2/x} \frac{d^n}{dx^n} (x^{2n} e^{-2/x})$$

Explicitly, we can obtain

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2}\right)^k$$

Let

$$H(n,k) = \frac{(n+k)!}{2^k (n-k)!k!}$$

Then,

$$y_n(x) = \sum_{k=0}^{n} H(n,k) x^k$$

It is easy to verify that

$$H(n+1,k) = H(n,k) + (n+k)H(n,k-1)$$

The polynomials $y_n(x)$ satisfy the recurrence relation

$$y_{n+1}(x) = (2n+1)xy_n(x) + y_{n-1}(x), \quad for \ \ n \geqslant \ 0$$

with initial conditions $y_{-1}(x) = y_0(x) = 1$. The first three Bessel Polynomials are expressed as

$$\begin{aligned} y_1(x) &= 1 + x, \\ y_2(x) &= 1 + 3x + 3x^2, \\ y_3(x) &= 1 + 6x + 15x^2 + 15x^3. \end{aligned}$$

We present here a grammatical characterization of the Bessel polynomials $y_n(x)$.

Proposition 4. *Let $G = \{I \to IX + IY; X \to 2X^2; Y \to XY\}$. Then, we have*

$$D^n(I) = I \sum_{k \geq 0} H(n,k) X^k Y^{n-k} = IY^n y_n(X/Y).$$

Note that $D^n(X) = n!2^n X^{n+1}$ and $D^n(Y) = (2n-1)!!X^n Y$. From Leibniz's formula, we obtain the following corollary:

Corollary 4. *For $n \geq 0$, we have*

$$y_{n+1}(x) = \sum_{k=0}^{n} \binom{n}{k} (2n-2k-1)!! y_k(x) x^{n-k} + \sum_{k=0}^{n} \frac{n!2^{n-k}}{k!} y_k(x) x^{n-k+1}.$$

3.4. The Exponential Riordan Array $[\exp(x/(1-x)), x/(1-x)]$

Definition 1 (see [16])**.** *The exponential Riordan group G is a set of infinite lower-triangular integer matrices, and each matrix in G is defined by a pair of generating function $g(x) = g_0 + g_1 x + g_2 x^2 + \cdots$ and $f(x) = f_1 x + f_2 x^2 + \cdots$, with $g_0 \neq 0$ and $f_1 \neq 0$. The associated matrix is the matrix whose i-th column has exponential generating function $g(x) f(x)^i / i!$ (columns marked from 0). The matrix corresponding to the pair f, g is defined by $[g, f]$.*

Let $R(n,k)$ be the (n,k)-th element in the matrix $[\exp(x/(1-x)), x/(1-x)]$. The associated Riordan array is given as follows:

$$\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 3 & 4 & 1 & & & \\ 13 & 21 & 9 & 1 & & \\ 73 & 136 & 78 & 16 & 1 & \\ \vdots & & & & & \ddots \end{pmatrix} \tag{6}$$

From A059110 [11], we see that

$$R(n,k) = \sum_{i=0}^{n} L'(n,i)\binom{i}{k}$$

for $0 \leq k \leq n$, where $L'(n,i) = \frac{n!}{i!}\binom{n-1}{i-1}$ are unsigned Lah numbers. It is routine to verify that

$$R(n+1,k) = R(n,k-1) + (n+k+1)R(n,k) + (k+1)R(n,k+1).$$

Hence, by Theorem 1, we obtain the following Proposition.

Proposition 5. *Let*

$$G = \{I \to IX + IY; X \to 2XY + Y^2; Y \to Y^2\}$$

Then, we have

$$D^n(I) = I\sum_{k\geq 0} R(n,k)X^kY^{n-k} := IY^n r_n(\frac{X}{Y}).$$

Note $D^n(X) = (n+1)!xY^n + nn!Y^{n+1}$, $D^n(Y) = n!Y^{n+1}$. From Leibniz's formula, we obtain the following corollary:

Corollary 5. *For $n \geq 0$, we have*

$$r_{n+1}(x) = \sum_{k=1}^{n}\binom{n}{k}(n-k+1)!r_k(x)(x+1)$$

In Table 1, we list some combinatorial sequences that satisfy (3). More examples can be found in similar tables in [17–19]. By using Theorem 1, we give the grammatical interpretation of the corresponding sequences, so that we can obtain more convolution formulas.

4. Real Rootedness

In this section, as an application, we will pay attention to the property of real roots of the row-generating functions in the array $\{B(n,k)\}_{0\leq k\leq n}$ in (3). For the sake of proving our results, some known results should be introduced beforehand.

Let $\{P_n(x)\}$ denote a Sturm sequence, which is a sequence of standard polynomials meeting the condition of $\deg P_n = n$ and $P_{n-1}(r)P_{n+1}(r) < 0$ whenever $P_n(r) = 0$ and $n \geq 1$. Let RZ represent the set of polynomials with only real roots. $\{P_n(x)\}$ is known as a generalized Sturm sequence (GSS) if $P_n \in RZ$ and zeros of $P_n(x)$ are separated by those of $P_{n-1}(x)$ for $n \geq 1$. As a special case of Corollary 2.4 in Liu and Wang [20] (also see Zhu, Yeh, and Lu [7]), the following result provides a unified method to many polynomials with only real zeros.

Table 1. Some combinatorial sequences satisfying formula (3).

$(a_1, a_2, a_3, b_1, b_2, b_3, c)$	Description	Entry
(1,−1,1,0,1,1,0)	Eulerian numbers	A173018
(2.−1,1,0,1,1,0)	Second-order Eulerian numbers	A008517
(0,1,0,0,1,0,0)	Surj(n,k)	A019538
(1,1,0,0,1,0,0)	Ward numbers	A134991
(0,0,1,0,1,0,0)	Stirling subset numbers	A008277
(0,0,−1,−1,−1,0,0)	Lah numbers $L_{n,k}$	A008297
(0,0,1,1,1,0,0)	Unsigned Lah numbers $L(n,k)$	A105278
(−2,1,−2,0,0,1,0)	Coefficients of Laguerre polynomials in reverse order	A021010
(0,0,1,0,0,1,0)	Binomial coefficients	A007318
(0,0,1,1,0,0,0)	Stirling cycle numbers	A132393
(0,0,1,−1,0,0,0)	Stirling numbers of the 1st kind $s(n,k)$	A008275
(0,0,1,0,1,2,1)	Production of the triangle of Stirling numbers of the 2nd kind with the Pascal's triangle read by rows	A137597
(0,0,1,0,1,0,1)	Set partitions without singletons	A217537
(0,0,1,0,2,1,2)	Exponential Riordan Array $[\exp(\sinh(x) * \exp(x)), \sinh(x) * \exp(x)]$	A154602
(0,1,0,0,2,1,1)	$n!\binom{n}{k}$	A196347
(0,1,0,0,2,2,1)	Row-generating function is $n!\sum_{k=0}^{n}\frac{(1+x)^{n-k}}{k!}$	A073474
(1,1,0,2,2,2,1)	The number of (n,k) labeled rooted Greg trees $(n \geq 1, 0 \leq k \leq n-1)$	A048160
(2,−1,2,0,0,0,1)	The number of fixed-point-free involutions of $1,2,\ldots,2n$ having k cycles with entries of opposite parities $(0 \leq k \leq n)$	A161119

Lemma 1. *Let $\{P_n(x)\}$ be a sequence of polynomials with nonnegative coefficients and* $0 \leq \deg P_n - \deg P_{n-1} \leq 1$. *Suppose that*

$$P_n(x) = (a_n x + b_n)P_{n-1}(x) + x(c_n x + d_n)P'_{n-1}(x)$$

where a_n, $b_n \in R$, and $c_n \leq 0$, $d_n \geq 0$. Then, $\{P_n(x)\}_{n\geq 0}$ is a generalized Sturm sequence.

For nonnegative array $B(n,k)$, which satisfies the recurrence relation (3), it is sufficient to assume that, for $n \geq 1$,

$$\begin{cases} a_1 n + a_2 k + a_3 - a_1 \geq 0 \quad \text{for} \quad 1 \leq k \leq n, \\ b_1 n + b_2 k + b_3 - b_1 \geq 0 \quad \text{for} \quad 0 \leq k \leq n-1, \\ c(k+1) \geq 0 \quad \text{for} \quad 0 \leq k \leq n-2, \end{cases}$$

which is equivalent to

$$\begin{cases} a_1 \geq 0, a_1 + a_2 \geq 0, a_2 + a_3 \geq 0, \\ b_1 \geq 0, b_1 + b_2 \geq 0, b_3 \geq 0, \\ c \geq 0. \end{cases}$$

Define $B_n(x) = \sum_{k\geq 0}^{n} B(n,k)x^k$ $(n \geq 0)$ as the row-generating functions of $B(n,k)$. Thus, $B_0(x) = 1$ and

$$B_1(x) = b_3 + (a_2 + a_3)x.$$

Moreover, it turns out that $B_n(x)$ follows from the recurrence relation (3) as

$$B_n(x) = [b_1n + b_3 - b_1 + (a_1n + a_2 + a_3 - a_1)x]B_{n-1}(x) + (c + b_2x + a_2x^2)B'_{n-1}(x),$$

which implies that

$$\deg(B_n(x)) - \deg(B_{n-1}(x)) \leq 1$$

for each n.

Theorem 2. *Let $\{B(n,k)\}_{n,k\geq 0}$ be the array defined in (3). Assume that $b_2 = a_2 + c$. Then, we have the following results:*

(i) *There exist polynomials $A_n(x)$ for $n \geq 0$ such that*

$$B_n(x) = a^n(1+x)^n A_n(\frac{d}{1+x}),$$

where $A_n(x)$satisfies the recurrence relation

$$\begin{aligned} A_n(x) \quad &= \tfrac{1}{a}\{(a_1+a_2)n + a_3 - a_1 + \tfrac{(b_1+c-a_1-a_2)n-c+b_3-b_1-a_3+a_1}{d}x\}A_{n-1}(x) \\ &+ \tfrac{x}{a}\{\tfrac{(a_2-c)x}{d} - a_2\}A'_{n-1}(x) \end{aligned} \tag{7}$$

with $A_0(x) = 1$, $a > 0$ and $d > 0$.

(ii) *Assume $b_1 \geq a_1$ and $b_3 \geq a_2 + a_3$. If $a_2 \leq 0$, then $\{B_n(x)\}_{n\geq 0}$ is a generalized Sturm sequence.*

Proof. (i) Because $b_2 = a_2 + c$, it is obvious that

$$B_n(x) = [b_1n + b_3 - b_1 + (a_1n + a_2 + a_3 - a_1)x]B_{n-1}(x) + (c + a_2x)(1+x)B'_{n-1}(x),$$

It can be proven that (i) holds by induction on n as follows.
As $n = 1$, we can obtain

$$\begin{aligned} A_1(x) &= \frac{1}{a}\{a_2 + a_3 + \frac{b_3 - a_2 - a_3}{d}x\} \\ B_1(x) &= b_3 + (a_2 + a_3)x. \end{aligned}$$

Thus, we have

$$B_1(x) = a(1+x)A_1(\frac{d}{1+x}).$$

By the induction hypothesis, it now turns out that

$$\begin{aligned} B'_{n-1}(x) =& a^{n-1}(n-1)(x+1)^{n-2}A_{n-1}(\frac{d}{1+x}) - a^{n-1}(x+1)^{n-1}A'_{n-1}(\frac{d}{1+x})\frac{d}{(1+x)^2} \\ =& \frac{(n-1)B_{n-1}(x)}{1+x} - da^{n-1}(x+1)^{n-3}A'_{n-1}(\frac{d}{1+x}). \end{aligned}$$

It follows from that recurrence relation (7) that, for $n \geq 2$,

$$\begin{aligned} & a^n(1+x)^n A_n(\frac{d}{1+x}) \\ =& \{((a_1+a_2)n + a_3 - a_1)(1+x) + (b_1 + c - a_1 - a_2)n - c + b_3 - b_1 - a_3 + a_1\}B_{n-1}(x) \\ & - (c + a_2x)(n-1)B_{n-1}(x) + (c + a_2x)(1+x)B'_{n-1}(x) = B_n(x) \end{aligned}$$

Thus, for $n \geq 1$, we can prove that

$$B_n(x) = a^n(1+x)^n A_n(\frac{d}{1+x}).$$

(ii) Evidently, in light of (i), $B_n(x)$ forms a generalized Sturm sequence if and only if (iff) $A_n(x)$ forms a generalized Sturm sequence. The nonnegativity of the coefficients for $A_n(x)$ needs to be considered firstly. Let $A_n(x) = \sum_{k=0}^{n} A(n,k)x^k$ for $n \geq 0$. Then, according to the recurrence relation (7), we obtain

$$\begin{aligned} A(n,k) =& \frac{(a_1+a_2)n - a_2k + a_3 - a_1}{a} A(n-1,k) \\ &+ \frac{(b_1+c-a_1-a_2)n - (c-a_2)k + b_3 - b_1 + a_1 - a_2 - a_3}{ad} A(n-1,k-1) \end{aligned}$$

for $n \geq 1$. Following from the nonnegativity of $\{B(n,k)\}_{n,k\geq 0}$, it holds

$$a_1 + a_2 \geq 0, a_1 \geq 0, a_2 + a_3 \geq 0$$

Furthermore, by the hypothesis condition, we obtain

$$\begin{cases} b_1 + c - a_1 - a_2 \geq c - a_2 \geq 0, \\ (b_1 + c - a_1 - a_2) - (c - a_2) = b_1 - a_1 \geq 0, \\ (b_1 + c - a_1 - a_2) - (c - a_2) + b_3 - b_1 + a_1 - a_2 - a_3 \geq 0. \end{cases}$$

Thus, $\{B(n,k)\}_{n,k\geq 0}$ is a nonnegative array. According to the recurrence relation (7) and Lemma 1, we can conclude that the polynomials $A_n(x)$ form a generalized Sturm sequence if $a_2 \leq 0$. For the same reason, the polynomials $B_n(x)$ form a generalized Sturm sequence. □

For example, the row-generating function of the Bell triangle $a(n,k)$ in Section 3 is $a_n(x) = \sum_{k=0}^{n} a(n,k)x^k$. Then, the polynomials satisfy

$$a_n(x) = (1+x)a_{n-1}(x) + (1+x)a'_{n-1}(x),$$

with $a_0(x) = 1$. Using Theorem 2 (i), there exists an array $A(n,k)$ such that

$$a_n(x) = \sum_{k=0}^{n} a(n,k)x^k = (1+x)^n A_n(\frac{1}{1+x})$$

where $A_n(x)$ for $n \geq 1$ satisfies the recurrence relation

$$A_n(x) = [(n-1)x+1]A_{n-1}(x) - x^2 A'_{n-1}(x)$$

where $A_0(x) = 1$ and $A_1(x) = 1$. Obviously, $A(n,k) = S(n,n-k)$ for $n \geq 1$. Applying Theorem 2 (ii), it can be proven that $\{a_n(x)\}$ for $n \geq 0$ is a generalized Sturm sequence.

Author Contributions: Methodology, R.R.Z. and J.Y.; validation, F.R.; writing, R.R.Z. All authors have read and agreed to the published version of the manuscript.

Funding: The first author was supported by the National Natural Science Foundation of China (NSFC No. 11501090) and the Natural Science Foundation of Hebei Province (A2019501024). The second author was supported by MOST 110-2115-M-017-002-MY2; The third author's research was partially supported by the National Natural Science Foundation of China (NSFC No. 61807029).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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